Building Gaussian Cluster States by Linear Optics

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The linear optical creation of Gaussian cluster states, a potential resource for universal quantum computation, is investigated. We show that for any Gaussian cluster state, the canonical generation scheme in terms of QND-type interactions, can be entirely replaced by off-line squeezers and beam splitters. Moreover, we find that, in terms of squeezing resources, the canonical states are rather wasteful and we propose a systematic way to create cheaper states. As an application, we consider Gaussian cluster computation in multiple-rail encoding. This encoding may reduce errors due to finite squeezing, even when the extra rails are achieved through off-line squeezing and linear optics.

Introduction.—The cluster-state model for quantum computation [1] is a conceptually interesting alternative to the more conventional circuit model [2]. Once a suitable multi-party entangled cluster state has been prepared, universal quantum gates can be effected through the cluster via only single-party projective measurements and feedforward. Though originally based upon qubits, the cluster-state model can be also applied to other discrete-variable systems (qudits) as well as to continuous quantum variables [3].

Linear optics represents one of the most practical approaches to the realization of quantum information protocols, both for discrete-variable (DV) [4] and continuous-variable (CV) implementations [5]. In the DV case, efficient entangling gates cannot be achieved with single photons and linear optics. Nonetheless, probabilistic gates can be applied off-line to an entangled multi-photon state that eventually serves as a resource for the on-line computation [6]. A similar approach uses DV photonic cluster states, leading to a significant reduction in the resource consumption [7, 8]. However, the generation of the optical cluster states remains highly probabilistic in this case.

Although up to six-qubit single-photon cluster states have been created via postselection using nonlinear and linear optics [9, 10], a possible deterministic, unconditional realization of optical cluster states would be based on continuous variables. Here, the resources are squeezed states of light and the Gaussian cluster states may be created via quadratic quantum nondemolition (QND) interactions [11]. These interactions, however, cannot be realized through beam splitters alone. Additional "on-line" squeezers are needed for every single link of the cluster state, again rendering the mechanism for cluster generation rather inefficient with current technology. Moreover, the squeezing of the resource states will always be finite, inevitably resulting in errors in the cluster computation. In this paper, we will address both issues: the avoidance of on-line squeezing in cluster-state generation and the reduction of finite-squeezing induced errors in cluster-state computation.

We will show that for any Gaussian cluster state, the canonical generation scheme in terms of QND-type interactions, can be entirely replaced by off-line squeezers and beam splitters. Moreover, we propose a systematic way on how to build alternative cluster-type states from potentially cheaper squeezing resources than needed for the canonical states. In any of these linear-optics schemes, the resource states require correspondingly more squeezing to compensate for the lack of extra squeezing in the beam-splitter network that replaces the QND coupling of the cluster nodes. Nonetheless, the main features of the canonical QND-made clusters can be preserved. As an example, we consider Gaussian cluster computation in multiple-rail encoding. This encoding may reduce errors caused by finite squeezing. We will see that multiple-rail encoding with linear-optics-made clusters, though requiring supposedly more off-line squeezing for the extra links of the larger clusters, can still lead to the same error reduction as for the canonical cluster states.

We define cluster-type states as those multi-mode Gaussian states for which certain quadrature correlations become perfect in the limit of infinite squeezing [11],

$$\left(\hat{p}_a - \sum_{b \in N_a} \hat{x}_b\right) \to 0, \quad a \in G. \tag{1}$$

Perfect correlations uniquely define the corresponding graph state (for the DV case, see [12]). Here we use the dimensionless "position" and "momentum" operators, $\hat{x}$ and $\hat{p}$, corresponding to the quadratures of an optical mode with annihilation operator $\hat{a} = \hat{x} + i\hat{p}$. The modes $a \in G$ correspond to the vertices of the graph, while the modes $b \in N_a$ are the nearest neighbors of mode a.

Canonical cluster states via linear optics.—The canonical way to build CV cluster states would be to send a number of single-mode squeezed states through a corresponding network of QND gates [3, 11]. Each individual QND gate could be realized via two beam splitters and a pair of on-line squeezers [13]. However, by including the initial single-mode squeezers into the QND network, the resulting total circuit corresponds to a big quadratic

Hamiltonian applied to a number of vacuum modes. This total transformation can be decomposed into a linear-optics circuit followed by single-mode squeezers and a second linear-optics circuit [13], where the first linear-optics circuit has no effect on the vacuum modes. Eventually one has just one linear circuit applied to a number of single-mode squeezed states; in principle, this works for any cluster (or graph) state. Let us explain this in a little more detail.

The canonical generation of CV cluster states [11] from momentum-squeezed vacuum modes via QND-type interactions can be described by $\hat{x}'_a = \hat{x}_a$ and $\hat{p}'_a = \hat{p}_a + \sum_{b \in N_a} \hat{x}_b$, where $\hat{x}_a = e^{+r}\hat{x}_a^{(0)}$ and $\hat{p}_a = e^{-r}\hat{p}_a^{(0)}$, $\forall a \in G$, with vacuum modes labeled by superscript (0). Note that, according to Eq. (1), the canonical cluster states become perfectly correlated for $r \to \infty$. We have the following linear Bogoliubov (LUBO) transformation,

$$\hat{a}'_a = \hat{x}'_a + i\hat{p}'_a \stackrel{!}{=} \sum_{l \in G} A_{al}\,\hat{a}_l^{(0)} + B_{al}\,\hat{a}_l^{(0)\dagger}. \quad (2)$$

Using $\hat{a}_l^{(0)} = \hat{x}_l^{(0)} + i\hat{p}_l^{(0)}$, we can extract the LUBO matrix elements for canonical cluster generation: $A_{aa} = \cosh r$, $B_{aa} = \sinh r$, $A_{ab} = B_{ab} = (i/2)e^{+r}$ ($\forall b \in N_a$), and $A_{al} = B_{al} = 0$ [$\forall l \notin N_a\,(l \neq a)$]. Now the input modes of the LUBO transformation are vacuum modes instead of the squeezed input modes of the QND network. The next step is to decompose this LUBO transformation into a linear-optics circuit $V^\dagger$, a set of single-mode squeezers, and another linear-optics circuit U [13]. Using the singular value decomposition for A and B, we have to satisfy $V = A^\dagger U A_D^{-1} = B^T U^* B_D^{-1}$ and $U = AVA_D^{-1} = BV^* B_D^{-1}$ for the unitary matrices U and V. Here A_D^2 is the diagonalized version of $AA^\dagger$ and $A^\dagger A$; similarly, we use B_D^2 for $BB^\dagger$ and $(B^\dagger B)^T$. The singular-value matrix equations lead to the conditions for U,

$$\text{Im}U_{al} - C_l(r)\sum_{b \in N_a}\text{Re}U_{bl} = 0\,, \quad \forall a, l \in G\,, \quad (3)$$

with

$$\text{Re}U_{al}\,[D_l(r) - M_{N_a}] - \sum_{b \in N_a}\sum_{k \in N_b, k \neq a}\text{Re}U_{kl} = 0\,, \quad (4)$$

where

$$C_l(r) \equiv \frac{\frac{e^{+r}}{2}\left(\sqrt{\lambda_l^A} + \sqrt{\lambda_l^B}\right)}{\sqrt{\lambda_l^A}\sinh r + \sqrt{\lambda_l^B}\cosh r}\,, \quad (5)$$

$D_l(r) \equiv (\lambda_l^B\cosh^2 r - \lambda_l^A\sinh^2 r)/[e^{+2r}(\lambda_l^A - \lambda_l^B)/4]$, and $\text{Re}U_{al} \equiv \text{Re}(U_{al})$, etc.; M_{N_a} represents the number of nearest neighbors of mode a. The double sum in Eq. (4) sums over all second neighbors of mode a (including multiple counting of identical neighbors of the nearest neighbors of mode a). The expressions $\sqrt{\lambda_l^A}$ are the singular
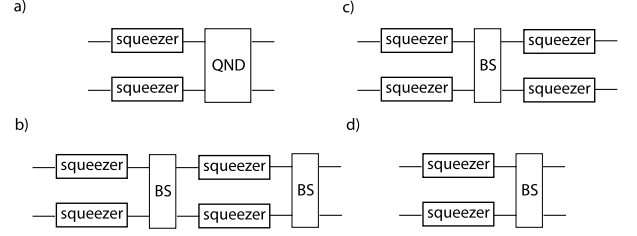

FIG. 1: Different generation schemes for a canonical two-mode cluster state. The input modes are always vacuum modes; a) canonical generation via QND interaction; b) QND gate replaced by a beam splitter (BS), two on-line squeezers, and another beam splitter [13]; c) generation of a standard two-mode squeezed state (via two squeezers and a beam splitter) plus two local squeezers [14]; d) simplest generation scheme with the QND gate replaced by a beam splitter.

values of A and similarly for B. Equation (3) can be incorporated into the column vectors of U,

$$\vec{u}_l \equiv \begin{pmatrix} \alpha_{1l} + iC_l(r)\sum_{b \in N_1}\alpha_{bl} \\ \alpha_{2l} + iC_l(r)\sum_{b \in N_2}\alpha_{bl} \\ \cdots \\ \alpha_{Nl} + iC_l(r)\sum_{b \in N_N}\alpha_{bl} \end{pmatrix}\,, \quad (6)$$

using $\text{Re}U_{kl} \equiv \alpha_{kl}$, with the constraints of Eq. (4), and $\sum_l (\vec{u}_l)_k(\vec{u}_l)^*_{k'} = \delta_{kk'}$ for unitarity. These conditions automatically satisfy $A = UA_DV^\dagger$ and $B = UB_DV^T$, where the diagonal matrices A_D and B_D contain the corresponding singular values. Thus, we effectively constructed a linear-optics circuit U that *exactly* outputs the canonical cluster states when applied to the *off-line* squeezed input modes with squeezed quadratures $\hat{x}_l|\hat{p}_l = \left(\sqrt{\lambda_l^A} \pm \sqrt{\lambda_l^B}\right)\hat{x}_l^{(0)}|\hat{p}_l^{(0)} \equiv e^{\pm R_l}\hat{x}_l^{(0)}|\hat{p}_l^{(0)}$. The vacuum modes, $\hat{a}_l^{(0)} = \hat{x}_l^{(0)} + i\hat{p}_l^{(0)}$, are "redefined" vacuum modes after the first linear-optics circuit $V^\dagger$ that has no effect.

Let us consider the example of the canonical two-mode cluster state. It corresponds to two momentum-squeezed modes (squeezed by r) coupled through a quadratic QND gate $e^{2i\hat{x}\otimes\hat{x}}$, see Fig.1a. In this case, we have $\lambda_l^A \equiv \cosh^2 r + e^{+2r}/4$, $\lambda_l^B \equiv \sinh^2 r + e^{+2r}/4$, $C_1(r) = C_2(r) \equiv C^{-1}(r)$, and $D_l(r) \equiv 1$. Thus, in the equivalent linear-optics scheme (Fig.1d), two equally squeezed modes are combined at a beam splitter described by U with column vectors as in Eq. (6), $l = 1, 2$, $N = 2$; the constraints in Eq. (4) are always satisfied.

A possible solution for U, choosing $\alpha_{12} = \alpha_{21} = 0$, is

$$U = \frac{1}{\sqrt{1 + C^2(r)}}\begin{pmatrix} C(r) & i \\ i & C(r) \end{pmatrix}. \quad (7)$$

Each input mode is momentum-squeezed with $R_l = \ln\left(\sqrt{\lambda_l^A} + \sqrt{\lambda_l^B}\right) \equiv R$. For $r = 0$, we obtain $\lambda_l^A \equiv 5/4$ and $\lambda_l^B \equiv 1/4$, and thus $R = \ln[(1 + \sqrt{5})/2]$. This is the

residual squeezing coming from the QND gate, which is now additionally applied off-line before the beam splitter. Note that the QND gate here is not simply replaced by a beam splitter followed by two single-mode squeezers and another beam splitter [13], all together applied to two initial squeezed states (Fig.1b). Instead, the circuit is further simplified, and only *one* pair of squeezers followed by *one* beam splitter operation act upon initial vacuum modes. However, in order to produce exactly the canonical state, this circuit is rather wasteful in terms of squeezing resources. The most economical two-mode entangled state is a standard two-mode squeezed state (TMSS) built from two single-mode squeezed states with a 50/50 beam splitter. Like any pure Gaussian two-mode state [14], also the canonical two-mode cluster state can be obtained from a TMSS via local Gaussian transformations, including local squeezers (Fig.1c).

Comparing resources, for example, an excess noise of one vacuum unit in the quadrature correlations $\hat{p}_1 - \hat{x}_2$ and $\hat{p}_2 - \hat{x}_1$ can be achieved with a TMSS built from two 3 dB squeezed states (see below). The canonical two-mode cluster state with these correlations ($r = 0$), would require two 4.18 dB squeezed states combined at the asymmetric beam splitter in Eq. (7). In general, the canonical N-mode cluster states are always biased in x and p [15], $\langle \hat{x}_a^2 \rangle = e^{+2r} \neq \langle \hat{p}_a^2 \rangle = e^{-2r} + M_{N_a} e^{+2r}$, unnecessarily requiring extra *local squeezing* to achieve a certain degree of correlations and entanglement; they cannot be obtained from N-mode pure Gaussian states in standard form [16] without the use of local squeezers. In the following, we shall consider a whole family of states producible via linear optics and exhibiting cluster-type correlations. This family will include states cheaper than the canonical cluster states.

Cluster-type states via linear optics.—In order to create states with correlations as in Eq. (1) via linear optics, we consider p-squeezed input modes, $\hat{a}_l = e^{+R_l} \hat{x}_l^{(0)} + i e^{-R_l} \hat{p}_l^{(0)}$, and a general linear-optics transformation, $\hat{a}_k' = \sum_l U_{kl} \hat{a}_l$, with a unitary matrix U. Using $\hat{a}_k' = \hat{x}_k' + i\hat{p}_k'$, one obtains the output quadrature operators, and in order to satisfy the correlations in Eq. (1), we assume the large noise terms (those proportional to e^{+R_l}) cancel. This is possible if and only if

$$\mathrm{Im} U_{al} - \sum_{b \in N_a} \mathrm{Re} U_{bl} = 0 \,, \quad \forall a, l \in G \,. \tag{8}$$

After inserting these conditions into the excess noise terms (those that vanish for infinite squeezing $R_l \to \infty$), we find that every quadrature correlation of Eq. (1) labeled by a has an excess noise variance

$$\sum_l \left[\mathrm{Re} U_{al} (1 + M_{N_a}) + \sum_{b \in N_a} \sum_{k \in N_b, k \neq a} \mathrm{Re} U_{kl} \right]^2 e^{-2R_l} \,, \tag{9}$$

times one unit of vacuum quadrature noise (1/4 in our scales) which we omit in the following. These excess noises will lead to the errors in the cluster computation [3]. Let us consider two-mode cluster-type states.

Look at the simple circuit described by

$$U = \begin{pmatrix} 1 & 0 \\ 0 & i \end{pmatrix} \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & i \end{pmatrix} = \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{i}{\sqrt{2}} \\ \frac{i}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix}, \tag{10}$$

applied to two momentum-squeezed modes. The circuit is a 50/50 beam splitter with Fourier transforms of mode 2 before and after the beam splitter. It clearly satisfies the conditions in Eq. (8). The resulting state is the standard TMSS up to a local Fourier transform of mode 2. No extra local squeezers are needed to obtain this cluster-type state from standard two-mode entanglement. According to Eq. (9), the excess noises in the cluster-type correlations are $2e^{-2R_1}$ in $\hat{p}_1 - \hat{x}_2$ and $2e^{-2R_2}$ in $\hat{p}_2 - \hat{x}_1$, corresponding to the optimal entanglement for a given input squeezing. Note that the solutions in Eq. (10) and Eq. (7) coincide for $r \to \infty$ when $C(r) \to 1$.

Incorporating Eq. (8) into the column vectors of U leads to

$$\vec{u}_l \equiv \begin{pmatrix} \alpha_{1l} + i \sum_{b \in N_1} \alpha_{bl} \\ \alpha_{2l} + i \sum_{b \in N_2} \alpha_{bl} \\ \cdots \\ \alpha_{Nl} + i \sum_{b \in N_N} \alpha_{bl} \end{pmatrix}, \tag{11}$$

with $\sum_l (\vec{u}_l)_k (\vec{u}_l)_{k'}^* = \delta_{kk'}$ for unitarity and again $\mathrm{Re} U_{kl} \equiv \alpha_{kl}$. Using this formalism, for any given graph state, one obtains a set of geometrical conditions for N real vectors $\vec{\alpha}_k^T \equiv (\alpha_{k1}, \alpha_{k2}, ..., \alpha_{kN})$. For instance, a linear 4-mode cluster state has $\vec{\alpha}_1 \vec{\alpha}_3 = \vec{\alpha}_2 \vec{\alpha}_4 = -1/5$, $\vec{\alpha}_1 \vec{\alpha}_2 = \vec{\alpha}_1 \vec{\alpha}_4 = \vec{\alpha}_2 \vec{\alpha}_3 = \vec{\alpha}_3 \vec{\alpha}_4 = 0$, $||\vec{\alpha}_1|| = ||\vec{\alpha}_4|| = 3/5$, and $||\vec{\alpha}_2|| = ||\vec{\alpha}_3|| = 2/5$, achievable, for example, with $\vec{\alpha}_1^T = (1/\sqrt{2}, 1/\sqrt{10}, 0, 0)$, $\vec{\alpha}_2^T = (0, 0, -2/\sqrt{10}, 0)$, $\vec{\alpha}_3^T = (0, -2/\sqrt{10}, 0, 0)$, and $\vec{\alpha}_4^T = (0, 0, 1/\sqrt{10}, -1/\sqrt{2})$. This, together with Eq. (11), where we set $N = 4$, defines a possible solution U. In general, any 4-mode transformation U can be decomposed into a network of $4(4-1)/2 = 6$ beam splitters [17]. However, for the linear 4-mode cluster, there is a minimal decomposition
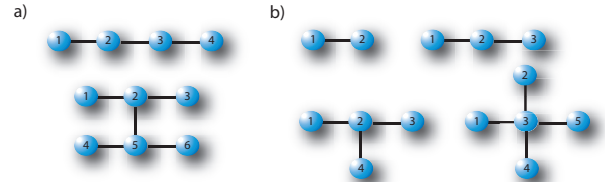

FIG. 2: Examples of cluster-type states producible via linear optics; each optical mode/cluster node a is represented by a vector $\vec{\alpha}_a$ (see text): a) linear 4-mode cluster, nonlinear 6-mode cluster; b) "EPR/GHZ-type" [18] clusters: linear 2 and 3-mode clusters, T-shape and cross-shape clusters.
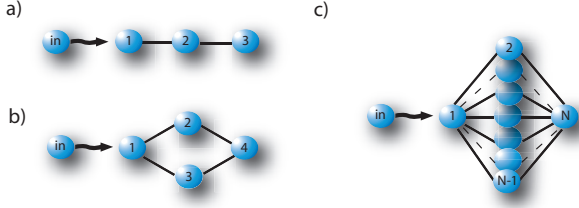
FIG. 3: Error propagation when an input state is attached to and teleported through a) a linear 3-mode cluster; b) a 4-mode diamond cluster; c) an N-mode multiple-rail cluster; all clusters are generated via off-line squeezing and linear optics.

of U into a network of only three beam splitters $U = F_4 S_{12} F_1^\dagger B_{34}^+(1/\sqrt{2}) B_{12}^+(-1/\sqrt{2}) B_{23}^-(1/\sqrt{5}) F_3^\dagger F_4^\dagger$. Here, F_k (Fourier transform of mode k) is the 4-mode identity matrix I_4 except $(F_k)_{kk} = i$, $B_{kl}^\pm(t)$ (beam splitter transformation of modes k and l) equals I_4 except $(B_{kl}^\pm)_{kk} = t$, $(B_{kl}^\pm)_{kl} = \sqrt{1-t^2}$, $(B_{kl}^\pm)_{lk} = \pm\sqrt{1-t^2}$, and $(B_{kl}^\pm)_{ll} = \mp t$, and the matrix S_{12} swaps modes 1 and 2.

Similarly, one can find linear-optics solutions for other cluster-type states (Fig.2). The example of a nonlinear 6-mode cluster in Fig.2a (a potential resource for *two-mode* evolutions) leads to two orthogonal subspaces $\{\vec{\alpha}_k, \vec{\alpha}_l, \vec{\alpha}_m\}$ with $\{k,l,m\} = \{1,3,5\}$ and $\{k,l,m\} = \{6,4,2\}$, where $\vec{\alpha}_k\vec{\alpha}_l = -3/10$, $\vec{\alpha}_k\vec{\alpha}_m = \vec{\alpha}_l\vec{\alpha}_m = -1/10$, and $||\vec{\alpha}_k|| = ||\vec{\alpha}_l|| = 7/10$, $||\vec{\alpha}_m|| = 3/10$. A possible solution is $(1/\sqrt{5}, 1/\sqrt{2}, 0, 0, 0, 0)$, $(1/\sqrt{5}, -1/\sqrt{2}, 0, 0, 0, 0)$, $(-1/\sqrt{20}, 0, 1/2, 0, 0, 0)$ for vectors $\{1,3,5\}$, respectively, and also $(0,0,0,-1/\sqrt{20}, 0, 1/2)$, $(0,0,0,1/\sqrt{5}, 1/\sqrt{2}, 0)$, $(0,0,0,1/\sqrt{5}, -1/\sqrt{2}, 0)$ for $\{2,4,6\}$. In general, using the geometrical conditions for N N-dimensional vectors, solutions can be recursively constructed for any graph state: start with $\vec{\alpha}_1^T = (\alpha_{11}, 0, 0, \ldots)$ determined by the norm $||\vec{\alpha}_1||$, then $\vec{\alpha}_2^T = (\alpha_{21}, \alpha_{22}, 0, \ldots)$ given by the overlap $\vec{\alpha}_1\vec{\alpha}_2$ and the norm $||\vec{\alpha}_2||$, $\vec{\alpha}_3^T = (\alpha_{31}, \alpha_{32}, \alpha_{33}, \ldots)$ via $\vec{\alpha}_1\vec{\alpha}_3$, $\vec{\alpha}_2\vec{\alpha}_3$, and $||\vec{\alpha}_3||$, etc.

Redundant encoding for error filtration.—As an application, we consider multiple-rail encoding in a Gaussian cluster computation protocol [3]. Look at the diamond cluster state in Fig.3b. In this case, the vector conditions become $\vec{\alpha}_1\vec{\alpha}_2 = \vec{\alpha}_1\vec{\alpha}_3 = \vec{\alpha}_2\vec{\alpha}_4 = \vec{\alpha}_3\vec{\alpha}_4 = 0$, $\vec{\alpha}_2\vec{\alpha}_3 = \vec{\alpha}_1\vec{\alpha}_4 = -2/5$, and $||\vec{\alpha}_k|| \equiv 3/5$. This can be satisfied, for example, via $\vec{\alpha}_1^T = (1/\sqrt{3}, -2/\sqrt{15}, 0, 0)$, $\vec{\alpha}_2^T = (0, 0, 1/\sqrt{3}, -2/\sqrt{15})$, $\vec{\alpha}_3^T = (0, 0, 0, \sqrt{3/5})$, and $\vec{\alpha}_4^T = (0, \sqrt{3/5}, 0, 0)$. A simple cluster protocol would be to teleport an input mode onto mode 1 of the diamond state and from there to mode 4 (Fig.3b). For simplicity, we assume that the input mode is attached to mode 1 through the QND gate $e^{2i\hat{x}_{\rm in}\otimes\hat{x}_1}$ such that $\hat{p}_1 \rightarrow \hat{p}_1 + \hat{x}_{\rm in}$ and $\hat{p}_{\rm in} \rightarrow \hat{p}_{\rm in} + \hat{x}_1$; $\hat{x}_1$ and $\hat{x}_{\rm in}$ remain unchanged. Now p-measurements of the input mode and modes 1 through 3 (with results $s_{\rm in}$, $s_{1\ldots3}$) would transform mode 4 into the input state up to some known corrections and excess noise coming from the im-

perfect diamond state. After the correction operation, $\hat{F}^\dagger \hat{X}(-s_{\rm in})\hat{F}^\dagger \hat{X}(-s_1)\hat{F}^\dagger \hat{X}(-(s_2+s_3)/2)$, with an inverse Fourier transform operator $\hat{F}^\dagger$ acting as $\hat{x} \rightarrow \hat{p}$ and $\hat{p} \rightarrow -\hat{x}$, and x-displacements $\hat{X}(s)$ such that $\hat{x} \rightarrow \hat{x}+s$, mode 4 is described by $\hat{x}_{\rm out} = \hat{x}_{\rm in}+(\hat{p}_1-\hat{x}_2-\hat{x}_3)-(\hat{p}_4-\hat{x}_2-\hat{x}_3)$ and $\hat{p}_{\rm out} = \hat{p}_{\rm in}-(\hat{p}_2-\hat{x}_1-\hat{x}_4)/2-(\hat{p}_3-\hat{x}_1-\hat{x}_4)/2$. Here, $\hat{p}_{\rm in}$ and $\hat{p}_1$ correspond to the momentum operators of the input mode and mode 1 before the QND coupling.

We see that the imperfect teleportation fidelities depend on the correlations of the diamond cluster, Eq. (1) with Fig.3b. In the corresponding excess noise variances, Eq. (9), let us consider only the effect of finite R_3 and R_4, assuming $R_1 \gg 1$ and $R_2 \gg 1$. This leads to negligible excess noises, $a = 1$ and $a = 4$, in $\hat{x}_{\rm out}$, because the vectors $\vec{\alpha}_1$ and $\vec{\alpha}_4$ live in the two-dimensional subspace $l = 1, 2$. In $\hat{p}_{\rm out}$, according to Eq. (9) with $a = 2$ and $a = 3$, we find a total excess noise of $(13\,e^{-2R_3} + 5\,e^{-2R_4})/12 = 3\,e^{-2R}/2$, assuming $R_3 = R_4 = R$, which is *half* the excess noise compared to *any* linear 3-mode cluster protocol (Fig.3a) with two modes highly squeezed and one finitely squeezed by R; the generality of this result can be easily proven using Eq. (9) and Eq. (11) for the 3-mode protocol, with $\hat{p}_{\rm out} = \hat{p}_{\rm in} - (\hat{p}_2 - \hat{x}_1 - \hat{x}_3)$ and $\hat{x}_{\rm out} = \hat{x}_{\rm in} + (\hat{p}_1 - \hat{x}_2) - (\hat{p}_3 - \hat{x}_2) \approx \hat{x}_{\rm in}$. More generally, we obtain a p excess noise of $3\,e^{-2R}/m$ for multiple-rail encoding (Fig.3c), where m is the number of intermediate nodes (number of rails) between mode 1 and the output mode [19]. As a result, the excess noise in cluster computation may be reduced by teleporting the input through multiple paths. Remarkably, the linear-optics-made clusters achieve the same error reduction as obtainable for QND-made clusters with QND gates freely available.

Conclusion.—We showed that any Gaussian cluster state can be built via off-line squeezing and linear optics without QND couplings. Simple vector conditions lead to potentially cheaper cluster-type states. In multiple-rail-encoded cluster computation, the same error reduction as for the QND-made clusters can be achieved. Our results pave the way for experimental realizations of small-scale cluster computation with continuous variables.

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[19] for the general case, the vector conditions are $\vec{\alpha}_1\vec{\alpha}_N = (m-2m^2)/(4m^2-1)$, $\vec{\alpha}_k\vec{\alpha}_l = (2-4m)/(4m^2-1)$, $(k, l \neq 1, N)$, $\vec{\alpha}_1\vec{\alpha}_k = \vec{\alpha}_N\vec{\alpha}_k = 0$, $||\vec{\alpha}_1|| = ||\vec{\alpha}_N|| = (2m^2 + m - 1)/(4m^2 - 1)$, and $||\vec{\alpha}_k|| = (4m^2 - 4m + 1)/(4m^2 - 1)$.